New class of Compactons in Generalized Korteweg-DeVries Equations and Global Relations

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Collaborators: Avinash Khare (Inst. of Physics, Bhubaneswar) and Avadh Saxena (LANL)

Original Interest in Solitary Wave dynamics

- Solitary waves appear in many nonlinear dynamical systems described by nonlinear partial differential equations.
- If we KNOW there are organized structures (solitary waves, vortices, blobs) can we determine their gross features without solving the equation (numerically or analytically)?
- – Led to try robust variational wave functions with time dependent parameters for width, Height and position of center.
- –Found universal relations between Height, Position, velocity, critical Mass

for self-focusing... seemed independent of choice of trial wave function, using time dependent variational principle.

- – Today will show that previous results can be DERIVED WITHOUT recourse to trial functions, using similar variational methods (Minimization of ACTION).
- Find analytically new two parameter class of compactons in Generalized K-dV equtaions

OUTLINE OF TALK

- We discuss two generalizations of the K-dV equation by Rosenau and Hyman (RH) and Cooper Shepard and Sodano (CSS)
- We find a new two parameter family of compact solitary wave solutions to both equations of the form
$$AZ^{\gamma}(\beta(x+ct)) \qquad (1)$$
, γ continuous, where
$$(Z')^2 = 1 - Z^{2q} \qquad (2)$$
and q is continuous. γ and q are related to the powers of nonlinearity in the equation of motion.
- We derive for the CSS equation an exact relation for all solitary wave solutions that the Height, Width and

velocity are related in a simple fashion.

- We explicity determine the Energy and Momentum for all these solutions and verify the relationship

$$E/P = c/r \tag{3}$$

for all the compacton solutions.

- We determine the domain of stability for the new solutions.

HISTORY OF COMPACTONS

- Discovered originally in extension of the KdV equation by Rosenau and Hyman [1].

$$K(m,n) : u_t + (u^m)_x + (u^n)_{xxx} = 0, \tag{4}$$

- $m = n \leq 3$ the solutions are

$$[\cos(\xi)]^{2/(m-1)}, \tag{5}$$

where $\xi = a(x - ct)$. $-\pi/2 \leq \xi \leq \pi/2$, zero elsewhere.

- We will show here that for $m = 2n - 1$ with n continuous variable $1 < n \leq 3$ t

$$u(x,t) = A\mathrm{cn}^{\gamma}(\beta(x - vt); k^2 = 1/2) \tag{6}$$

- We then will find solutions for *ALL* m, n with $1 < n \leq 3$ and m, n continuous.
- The RH equations were not derivable from a Hamiltonian.
- Cooper, Shepard and Sodano (CSS) [3] considered instead

$$K^*(l,p): \quad u_t = u_x u^{l-2} + \alpha\,(2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}(u_x)^3) \quad (7$$

Lagrangian Hamiltonian Dynamics $\rightarrow$ VARIATIONAL FORMULATION

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x \varphi_t - \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p (\varphi_{xx})^2 \right) dx, \quad (8)$$

- SAME class of solitary wave solutions when $l = m+1$ and $p = n-1$.
- Using TRIAL WAVE FUNCTIONS

$$u_v(x,t) = A(t) \exp\left[-\beta(t)|x+q(t)|^{\gamma}\right] \quad (9)$$

$$\dot{q} = r(p,l)\frac{E}{P} \quad (10)$$

$r(p,l) = (p+l+2)/(p+6-l)$.

- When $l = p+2$ the width did not depend on the amplitude or velocity

- We will show using Hamilton's equations this result is exact for

$$u(x,t) = AZ[\beta(x + q(t))] \qquad (11)$$

Solitary Waves in Rosenau Hyman equation

- Let $u = f(y)$ where $y = x - vt$

$$vf' = \frac{d}{dy}(f^m) + \frac{d^3}{dy^3}(f^n). \qquad (12)$$

- Integrating twice

$$\begin{aligned} \frac{n}{n+1} v f^{n+1} &= \frac{n}{n+m} f^{n+m} + \frac{1}{2}[\frac{d(f^n)}{dy}]^2 \\ &+ C_1 f^n + C_2 \qquad (13) \end{aligned}$$

- compactons are solutions with $C_1 = C_2 = 0$.

Circular Function solutions.

- RH equation

$$(f')^2 = \frac{2v}{n(n+1)} f^{3-n} - \frac{2}{n(n+m)} f^{m-n+2}. \tag{14}$$

- Choose Ansatz for $m = n$

$$f = A\cos^{2/(m-1)}(\beta(x - vt)); \tag{15}$$

$-\pi/2 \leq \beta y \leq \pi/2.$, and $f = 0$ elsewhere. $(m, n) = (2, 2)$

$$\beta = 1/4; \quad A = \frac{4}{3}v \tag{16}$$

- For $m = n$, $\beta = constant$, independent of the Amplitude which depends on v

Elliptic function solutions of RH equation.

- Solutions of the form cn^m.
- $(m,n)=(3,2)$

$$\frac{1}{3}vf-\frac{1}{5}f^3=(f')^2 \tag{17}$$

$f=A\mathrm{cn}^2(\beta(x-vt);k^2)$.

$$A=10\beta^2;,\quad k^2=1/2;\beta^4=\frac{v}{60}. \tag{18}$$

- $(m,n)=(5,3)$.

$$\frac{v}{2}-\frac{1}{4}f^4=3(f')^2. \tag{19}$$

$f=A\mathrm{cn}(\beta y;k^2)$,

$$A^2=6\beta^2;k^2=1/2\ \ ;\beta=(\frac{v}{18})^{1/4}. \tag{20}$$

New Class of Elliptic Solutions

- Find $k^2 = 1/2$ is special,

$$\mathrm{dn}^2(x, k^2 = 1/2)\mathrm{sn}^2(x, k^2 = 1/2)$$
$$= \frac{1}{2}(1 - \mathrm{cn}^4(x, k^2 = 1/2)) \qquad (21)$$

$$f = A\mathrm{cn}^\gamma(\beta y, k^2 = 1/2) \qquad (22)$$

$$\begin{aligned} m &= 2n - 1 \\ \gamma &= 2/(n-1) \\ A^{2n-2} &= \frac{3n-1}{n+1}v \\ \gamma^4\beta^4 &= \frac{16v}{n^2(3n-1)(n+1)}. \end{aligned} \qquad (23)$$

- To prevent singular solutions

$$1 < n \leq 3. \qquad (24)$$

- Case $(m,n)=(4,2)$

$$(f')^2=\frac{vf}{3}-\frac{f^4}{6} \tag{25}$$

- Put in standard Form:

$$f=AZ^2(\beta y) \tag{26}$$

$(\xi=\beta y)$

$$(\frac{dZ}{d\xi})^2=\frac{2v-A^3Z^6}{24A\beta^2} \tag{27}$$

- Choose

$$A=(2v)^{1/3};\quad \beta=\frac{(2v)^{1/3}}{2\sqrt{6}} \tag{28}$$

$$\pm\int_0^Z \frac{dz}{\sqrt{1-z^6}}=\xi;\ 0\le Z\le 1. \tag{29}$$

• Simplifying:

$$Z(\xi) = \left[\frac{1 - \mathrm{cn}(2(3)^{1/4}\xi)}{(1+\sqrt{3}) + (\sqrt{3}-1)\mathrm{cn}(2(3)^{1/4}y)}\right]^{1/2} \tag{30}$$

$$k^2 = \frac{1}{2} - \frac{\sqrt{3}}{4}. \tag{31}$$

- NEW Solutions ANSATZ: FIND values of a and m, n so that

$$f = AZ^{a}(\beta y) \tag{32}$$

leads to the differential equation

$$\left(\frac{dZ(\xi)}{d\xi}\right)^2 = 1 - Z^{2q}(\xi) \tag{33}$$

Here we allow for m, n, q to be continous variables.

- Conditions are

$$\begin{aligned} & m = q(n-1) + 1; \quad a = 2/(n-1) \\ & A^{m-1} = v\frac{n+m}{n+1}; \\ & a^2\beta^2A^2 = \frac{2}{n(n+m)}A^{m-n+2}. \end{aligned} \tag{34}$$

$$\beta^2 = \frac{(n-1)^2}{2(n+m)}\left[\frac{(n+m)v}{n+1}\right]^{(m-n)/(m-1)} \tag{35}$$

$1 < n \leq 3$.

- Special solutions: $q = 1$ $(m = n)$, circular compactons with width independent of *velocity*, $q = 2$ solutions of the form $\mathrm{cn}^{\gamma}(k^2 = 1/2)$, solutions with $q = 3$, leading to Weirstrass functions.

- we will show that stability requires

$$(n-1)(q-1) < 4 \tag{36}$$

CSS equations

- CSS equation of motion

$$K^*(l,p): u_t = u_x u^{l-2}$$
$$+\alpha \Big(2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}$$

- can be derived from $\int L(x,t), dxdt$ (Least Action)

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x\varphi_t - \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p(\varphi_{xx})^2\right) \tag{38}$$

$$u(x) = \varphi_x(x). \tag{39}$$

RH set (m,n) corresponds to the CSS set $(l-1, p+1)$.)

SOLITARY WAVES

- Solitary Wave Ansatz:

$$u(x,t) = f(y) = f(x+ct), \quad (40)$$

$$\begin{aligned} cf' &= f'f^{l-2} + \alpha \Big[2f'''f^p + 4pf^{p-1}f'f'' \\ &\quad + p(p-1)f^{p-2}f'^3 \Big]. \end{aligned} \quad (41)$$

- Integrate Twice

$$\frac{c}{2}f^2 - \frac{f^l}{l(l-1)} - \alpha f'^2 f^p = C_1 f + C_2. \quad (42)$$

C_1 and C_2 are zero for compactons Well behaved : : $l > 1$ and $f'' f^p \to 0$, $f'^2 f^{p-1} \to 0$ at edges where $f \to 0$.

- CSS equation

$$\alpha f'^2 = \frac{c}{2} f^{2-p} - \frac{f^{l-p}}{l(l-1)}. \qquad (43)$$

For finite f' at the edges, we must have $p \leq 2, l \geq p$.

- Compare RH equation

$$(f')^2 = \frac{2v}{n(n+1)} f^{3-n} - \frac{2}{n(n+m)} f^{m-n+2} \qquad (44)$$

$l = m + 1$ and $p = n - 1$ equations identical in form, differing coefficients.

Special Cases and Relations:

- $l = p + 2$ $(m = n)$

$$u(x,t) = A\cos^{2/p}[\beta(x+ct)] \quad (45)$$

for noninteger p.

$$E = \frac{2P}{p+2}\dot{q}. \quad (46)$$

- $l = 2p + 2$

$$f = A\mathrm{cn}^{\gamma}(\beta y; k^2 = 1/2) \quad (47)$$

$$\alpha(f')^2 = \frac{c}{2}f^{2-p} - \frac{f^{p+2}}{(2p+2)(2p+1)}. \quad (48)$$

$$\gamma = \frac{4}{l-2};$$

$$\frac{c}{2} = \frac{A^{l-2}}{l(l-1)}$$
$$\beta^4\alpha^2 = \frac{2c}{l(l-1)}(\frac{l-2}{4})^2 \quad (49)$$

- Nonsingular behavior condition:

$$2 < l \leq 6. \quad (50)$$

- The rest of the story:

$$f = AZ^a(\xi = \beta y) \quad (51)$$
$$(Z')^2 = 1 - Z^{2q} \quad (52)$$

- This leads to continuous q

$$l = pq + 2; \quad a = 2/p$$
$$A^{l-2} = l(l-1)\frac{c}{2}; \quad \alpha a^2\beta^2 A^2 = \frac{c}{2}A^{2-p}. \quad (53)$$

$$\beta^2 = \frac{c}{2\alpha a^2 A^p} \quad (54)$$

- For well behaved solutions we need $0 < p \leq 2$.
- We will show

$$E/P = c/r \tag{55}$$

and stable for

$$p(q-1) < 4 \tag{56}$$

Conservation laws and canonical structure

- Canonical form for KdV

$$u_t = \partial_x \frac{\delta H}{\delta u} = \{u, H\} \tag{57}$$

$$\begin{aligned} H &= \int [(\pi\dot{\varphi}) - L]\, dx \\ &= \int \left[\frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p(\varphi_{xx})^2\right] dx, \\ &= \int \left[\frac{u^l}{l(l-1)} - \alpha u^p (u_x)^2\right] dx. \quad (58) \\ & \qquad (59) \end{aligned}$$

- Poisson bracket structure

$$\{u(x), u(y)\} = \partial_x \delta(x - y). \tag{60}$$

- Conservation of Mass

$$u_t = \partial_x \frac{\delta H}{\delta u} \qquad (61)$$

$$M = \int u(x,t)dx \qquad (62)$$

- Momentum Conservation Multiply eq. 61 by $u(x,t)$

$$\partial_t(\frac{u^2}{2}) = \partial_x \left[\frac{u^l}{l} + \alpha\{(p-1)u^p u_x^2 + 2u^{p+1}u_{xx}\} \right. \qquad (63)$$

$$(1/2) \int u^2(x,t)dx = P \qquad (64)$$

- P is the generator of the space translations:

$$\{u(x,t), P\} = \frac{\partial u}{\partial x}. \qquad (65)$$

- (i) $\phi(x,t) \longrightarrow \phi(x,t) + c_1$; (ii) $x \longrightarrow x + c_2$ nd (iii) $t \longrightarrow t + c$.

Energy-Momentum relationship

- ACTION

$$\Gamma = \int L dt, \tag{66}$$

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x \varphi_t - \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p (\varphi_{xx})^2 \right) \tag{67}$$

- Generic Solitary Wave

$$\phi_x = AZ(\beta(x + q(t))) = u, \tag{68}$$

- Using

$$\phi_t = \phi_x \dot{q}. \tag{69}$$

$$\int \frac{1}{2}\varphi_x \varphi_t dx = P\dot{q} \tag{70}$$

$$P = \frac{1}{2} \int u^2(x,t) dx \tag{71}$$

• Now have point "Particle" Lagrangian

$$L = P\dot{q} - H \tag{72}$$

$$H = \int dx \left[\frac{u^l}{l(l-1)} - \alpha u^p(\beta x)(u_x)^2 \right] \tag{73}$$

• Using

$$u_x = \beta A Z'[\beta(x + q(t))] \tag{74}$$

$$H = C_1(l) \frac{A^l}{\beta l(l-1)} - \alpha\beta A^{p+2} C_2(p) \tag{75}$$

where

$$C_1(l) = \int Z^l(z)dz; \quad C_2(p) = \int [Z'(z)]^2 Z^p(z)dz \tag{76}$$

• Since H is independent of q,

$$\dot{P} = -\frac{\partial H}{\partial q} = 0, \tag{77}$$

P is conserved.

- Rewrite A in terms of P :

$$P = \frac{1}{2} \int dx u^2 = \frac{A^2}{2\beta} C \tag{78}$$

$$C = \int dz Z^2(z) \tag{79}$$

$$A^2 = \frac{2\beta P}{C} \tag{80}$$

$$H = C_3(l) P^{l/2} \beta^{(l-2)/2} - C_4(p) P^{(p+2)/2} \beta^{(p+4)/} \tag{81}$$

where

$$C_3(l) = \frac{C_1(l)}{l(l-1)} [\frac{2}{C}]^{l/2}; \;\; C_4 = \alpha C_2(p) [\frac{2}{C}]^{(p+2)/2} \tag{82}$$

- Key Point: exact solutions minimize the Hamiltonian with respect to β .

$$\frac{\partial H}{\partial \beta} = 0, \tag{83}$$

$$\beta = P^{\frac{p-l+2}{l-p-6}} \left[\frac{C_4}{C_3}\frac{p+4}{l-2}\right]^{2/(l-p-6)}. \tag{84}$$

NOTE if $p = l - 2$ β INDEPENDENT OF P

ELIMINATING β

$$H = f(l,p)P^r \tag{85}$$

$$r = \frac{p+l+2}{p+6-l} \tag{86}$$

$$\dot{q} = \frac{\partial H}{\partial p} = r\frac{H}{P} \tag{87}$$

Calculating H and P exactly!!

- use the equation of motion for the solitary waves

$$H = \int dx \left[\frac{2f^l}{l(l-1)} - \frac{c}{2} f^2 \right] \quad (88)$$

- Use exact solution $f = AZ^a(\xi = \beta y)$, with $A^{l-2} = l(l-1)c/2$

$$H = \frac{A^2 c}{2\beta} \int d\xi [2Z^{a(pq+2)}(\xi) - Z^{2a}(\xi)] \quad (89)$$

$$P = \frac{A^2}{2\beta} \int d\xi Z^{2a}(\xi) \quad (90)$$

- Use the equation for Z to change variables from ξ to Z

$$dZ/d\xi = \sqrt{1 - Z^{2q}} \quad (91)$$

$$H = \frac{A^2 c}{\beta} \int_0^1 \frac{dZ}{\sqrt{1 - Z^{2q}}} [2Z^{a(pq+2)} - Z^{2a}] \tag{92}$$

- Evalueat in terms of the Beta function $\mathrm{B}(\mu, \nu)$ by substituting $t = Z^{2q}$.

$$H = \frac{A^2 c}{2\beta q} \frac{(6 + p - l)}{(l + p + 2)} \mathrm{B}\left(\frac{p+4}{2pq}, \frac{1}{2}\right). \tag{93}$$

$$P = \frac{A^2}{2\beta q} \mathrm{B}\left(\frac{p+4}{2pq}, \frac{1}{2}\right). \tag{94}$$

Using $a = 2/p$, and $a(l-2) = 2q$

$$H/P = c/r \tag{95}$$

Stability of Solutions

- The stability problem at $q = 1$ was studied by Dey and Karpman.

- The result of detailed analysis is that the criteria for Linear Stability is equivalent to the condition,

$$\frac{\partial P}{\partial c} > 0. \tag{96}$$

$$P = \frac{A^2}{2\beta q} \mathrm{B}\left(\frac{p+4}{2pq}, \frac{1}{2}\right). \tag{97}$$

$$A^{l-2} = l(l-1)\frac{c}{2}; \;\; \alpha a^2 \beta^2 A^2 = \frac{c}{2} A^{2-p}. \tag{98}$$

- Deduce

$$p(q-1) < 4 \tag{99}$$

The requirement for non-singular solutions is that $0 < p \leq 2$.

$$0 < p < 4/(q-1). \tag{100}$$

Analysis of Lyapunov stability following [10] [11] [12] leads to the same restrictions on p.

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