Mapping nonlinear turbulence to linear models: mechanism for anomalous scaling

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The problem:

In turbulent systems one can measure the velocity field .1 $\mathbf{u}(\mathbf{r}, t)$

2. When the turbulence is homogeneous (in space) and stationary (in time) one can define time-independent structure functions.

$$S_p(\mathbf{R}) \equiv \langle \left[(\mathbf{u}(\mathbf{x}+\mathbf{R}, t) - \mathbf{u}(\mathbf{x}, t)) \cdot \hat{\mathbf{R}} \right]^p \rangle_{t,x}$$

3. One observes that the structure functions are homogeneous function of their argument

$$S_p(\lambda \mathbf{R}) = \lambda^{\zeta_p} S_p(\mathbf{R})$$

4. Dimensional analysis predicts $\zeta_p = p/3$

5. In reality $\zeta_p \neq p/3 \text{ for all } p \neq 3$

In one family of problems the issues were resolved:

Turbulent Advection

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{f},$$

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\theta = \kappa \Delta \theta + f.$$

Turbulent advection is a linear problem

$$\frac{\partial \theta(\boldsymbol{r},t)}{\partial t} = \mathcal{L}\theta(\boldsymbol{r},t) + f(\boldsymbol{r},t)$$

$$\mathcal{L} \equiv \boldsymbol{u}\cdot\boldsymbol{\nabla} - \kappa \Delta.$$

Associated with the forced problem there is a decaying problem

$$\partial \theta / \partial t = \mathcal{L}\theta,$$

$$\theta(\boldsymbol{r},t) = \int d\boldsymbol{r}' R(\boldsymbol{r},\boldsymbol{r}',t)\theta(\boldsymbol{r}',0),$$

$$R(\boldsymbol{r},\boldsymbol{r}',t) \equiv T^{+}\exp\left[\int_0^t ds\, \mathcal{L}(s)\right]\Big|_{\boldsymbol{r},\boldsymbol{r}'}$$

The statistical theory of turbulent advection: anomalous scaling

$$C^{(N)}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N,t)\equiv\langle\theta(\boldsymbol{r}_1,t)\cdots\theta(\boldsymbol{r}_N,t)\rangle$$

$$C^{(N)}(\underline{\boldsymbol{r}},t)=\int d\underline{\boldsymbol{\rho}}\mathcal{P}^{(N)}(\underline{\boldsymbol{r}},\underline{\boldsymbol{\rho}},t)C^{(m)}(\underline{\boldsymbol{\rho}},0)$$

$$\mathcal{P}^{(N)}(\underline{\boldsymbol{r}},\underline{\boldsymbol{\rho}},t)\equiv\langle R(\boldsymbol{r}_1,\boldsymbol{\rho}_1,t)R(\boldsymbol{r}_2,\boldsymbol{\rho}_2,t)\cdots R(\boldsymbol{r}_N,\boldsymbol{\rho}_N,t)\rangle$$

The key finding is that these operators have eigenfunctions of eigenvalue 1

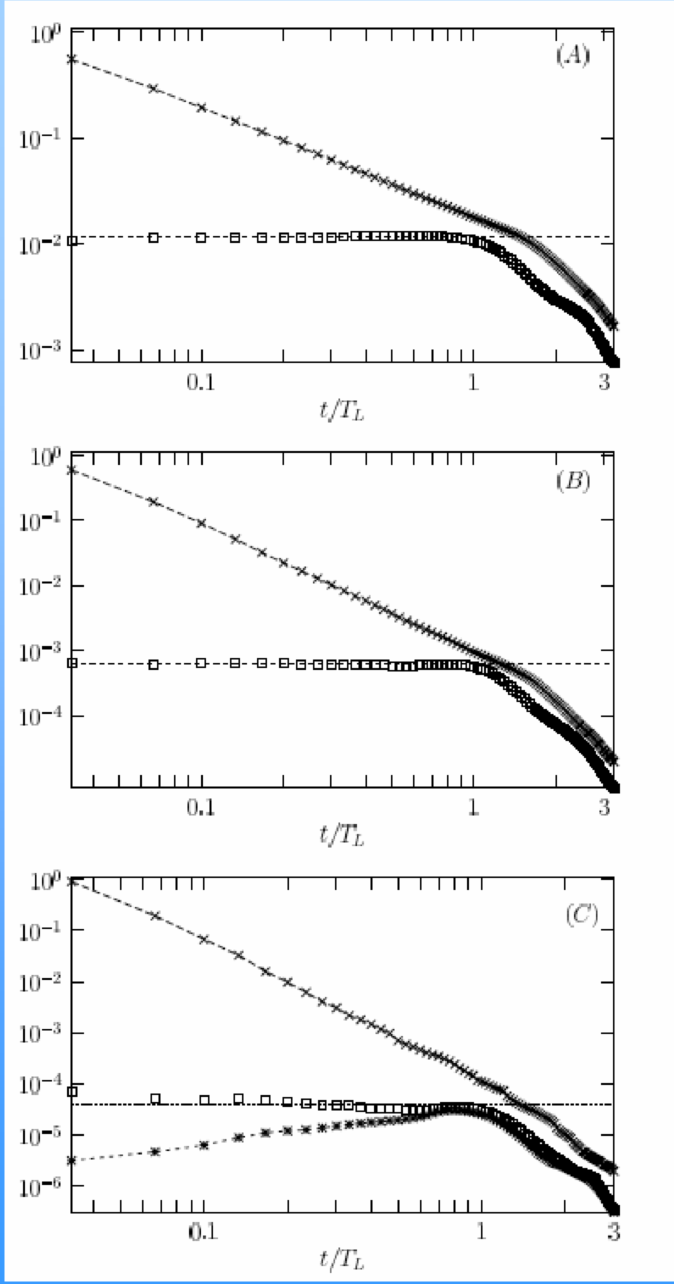
(I. Arad et al, PRL **87**, 164502 (2001), Y. Cohen et al, PRE **68**, 036303 (2003))

$$Z^{(N)}(\underline{\boldsymbol{r}})=\int d\underline{\boldsymbol{\rho}}Z^{(N)}(\underline{\boldsymbol{\rho}})\mathcal{P}^{(N)}(\underline{\boldsymbol{r}},\underline{\boldsymbol{\rho}},t)$$

$$Z^{(N)}(\lambda\underline{\boldsymbol{r}})=\lambda^{\zeta_N}Z^{(N)}(\underline{\boldsymbol{r}})+\cdots,$$

Existence of infinitely many statistically conserved variables

$$I_n\equiv\int d\underline{\mathbf{r}}Z^{(N)}(\underline{\mathbf{r}})C^{(N)}(\underline{\mathbf{r}},t)=$$
$$=\int\int d\underline{\mathbf{r}}d\underline{\mathbf{r}}'Z^{(N)}(\underline{\mathbf{r}})\mathcal{P}^{(N)}(\underline{\mathbf{r}},\underline{\mathbf{r}}',t)C^{(N)}(\underline{\mathbf{r}}',0)\ .$$



(A)
(B)
(C)
t/T_L
t/T_L
t/T_L

The nonlinear problem

L. Angheluta, R. Benzi, L. Biferale, I. Procaccia and F. Toschi, (cond-matt/0604050)

The key idea: construct a linear model whose scaling exponents are the same as the nonlinear problem

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \lambda \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f},$$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{w} + \lambda \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{w} = -\boldsymbol{\nabla} \tilde{p} + \nu \nabla^2 \boldsymbol{w} + \tilde{\boldsymbol{f}},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{w} = 0$$

For $\lambda = 0$. The first equation is Navier-Stokes and the second is “passive vector with pressure”. The latter is known to have SPS with anomalous scaling

(I Arad and IP, PRE 63, 056302 (2001)).

For $0 < \lambda < \infty$ the two fields must have the same exponents because of the symmetry $\lambda \boldsymbol{w} \leftrightarrow \boldsymbol{u}$

Consider the two composite fields $\boldsymbol{u}_+ \equiv \boldsymbol{u} + \lambda \boldsymbol{w}$ and $\boldsymbol{u}_- \equiv \boldsymbol{u} - \lambda \boldsymbol{w}.$

The first satisfies exactly the N-S equation, and the second satisfied exactly the passive vector equation.

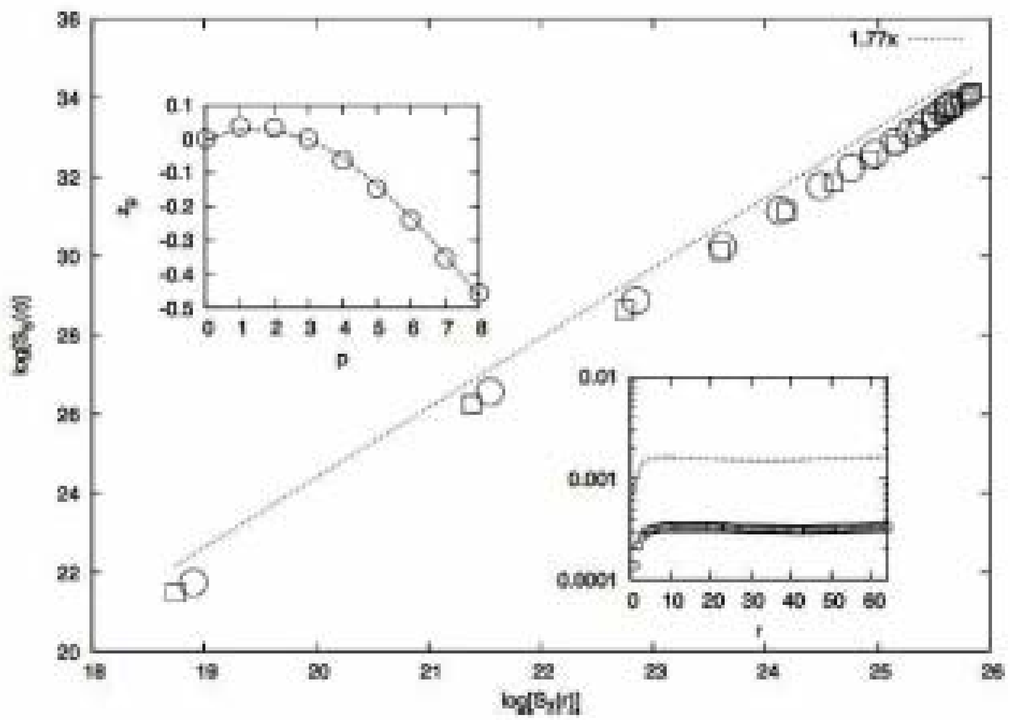


FIG. 1: Log-log plot of the sixth order structure functions of the fields u_+ and u_- (circles and squares respectively), for $\lambda = 1$, as a function of the third order structure functions. The dashed line corresponds to the best fit in the scaling region with slopes 1.77. Lower insert: $S_6(r)/S_3(r)^{1.77}$. Upper insert: $z_p \equiv \zeta_p/\zeta_3 - p/3$ computed for the structures functions of u_+ (line) and u_- (circles).

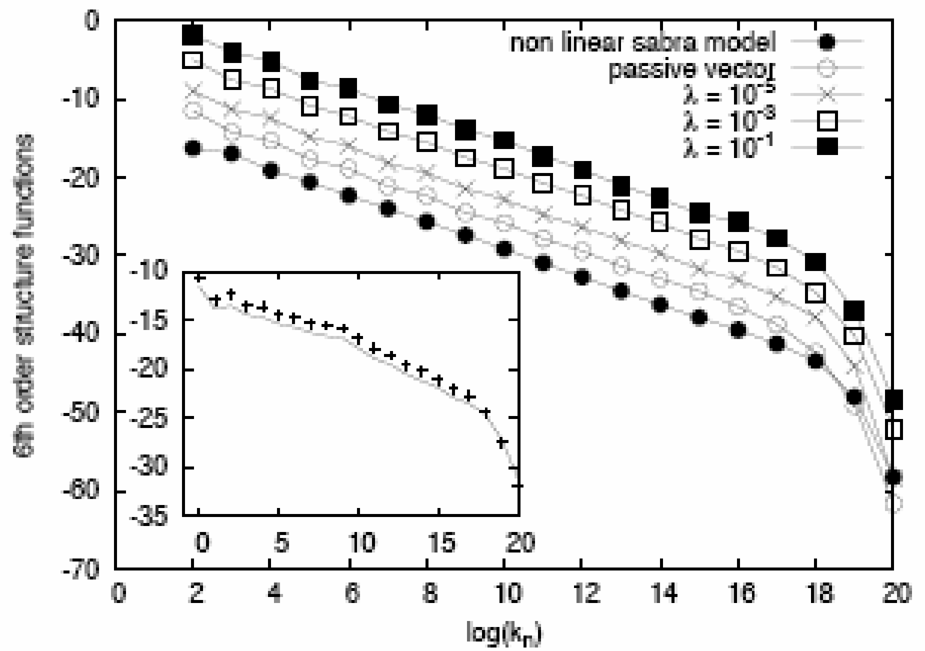


FIG. 2: The sixth order structure function of the field w_n in Eqs. (10) for $\lambda = 10^{-1}, 10^{-3}$ and 10^{-5}, together with the sixth order structure function for the Sabra model (2) and for the linear model (5), respectively. The structure function of the field u_n for $\lambda > 0$ are not shown since they are indistinguishable from those of the w_n. Inset: log-log plot of the fourth-order correlation function $F_{2,2}(k_n, k_7)$ vs. k_n calculated for the linear field (+) and for the nonlinear field (solid line) at $\lambda = 0$.

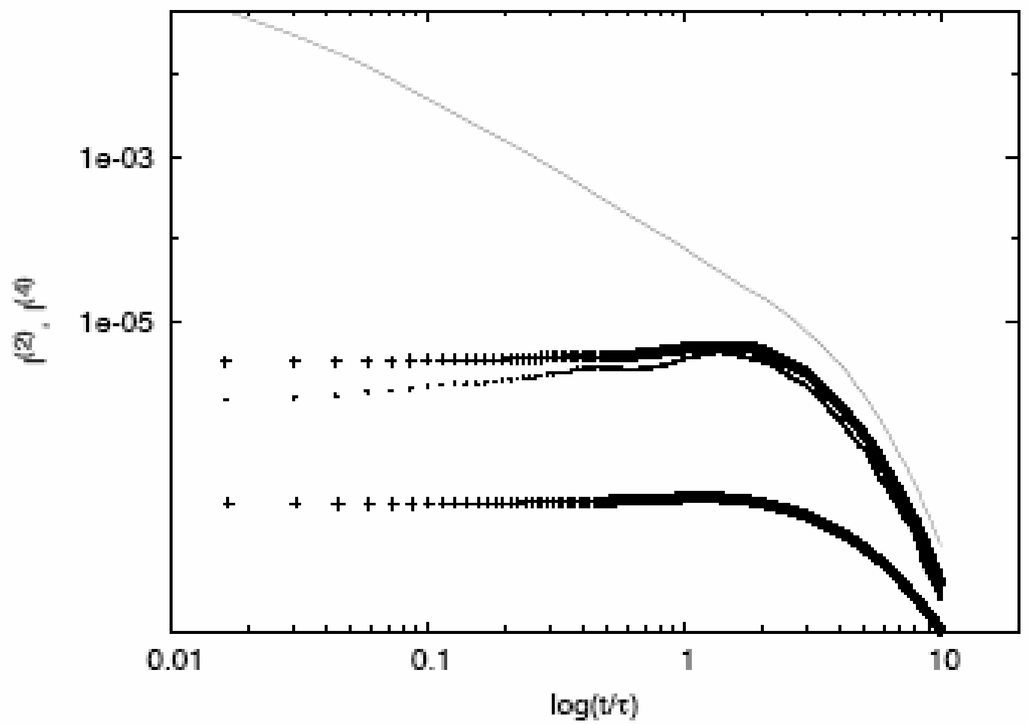


FIG. 3: With the symbols (+) the constants $I^{(2)}$ (bottom) and $I^{(4)}$ (top) constructed by projecting the decaying structure function of the linear model on the *forced* structure function of the **nonlinear** model. To emphasize the importance of using the correct SPS, we also show the result for $I^{(4)}$ using the dimensional Kolmogorov prediction for Z^4 (small dots) and $Z^4 = 1$ (solid line)

Summary

Anomalous scaling in turbulent advection is due to the (1 SPS of the advection propagator

2)For every nonlinear turbulent problem we can construct a linear advection model with the same exponents

3) Even though the nonlinear problem does not have a propagator, we find that the anomalous exponents are determined nevertheless by the 1-eigenfunctions of the advection operator.