On Formulating Continuum Mechanics of Fractal Media

Martin Ostoja-Starzewski
Mechanical Science & Engineering
UIUC

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1 Motivation

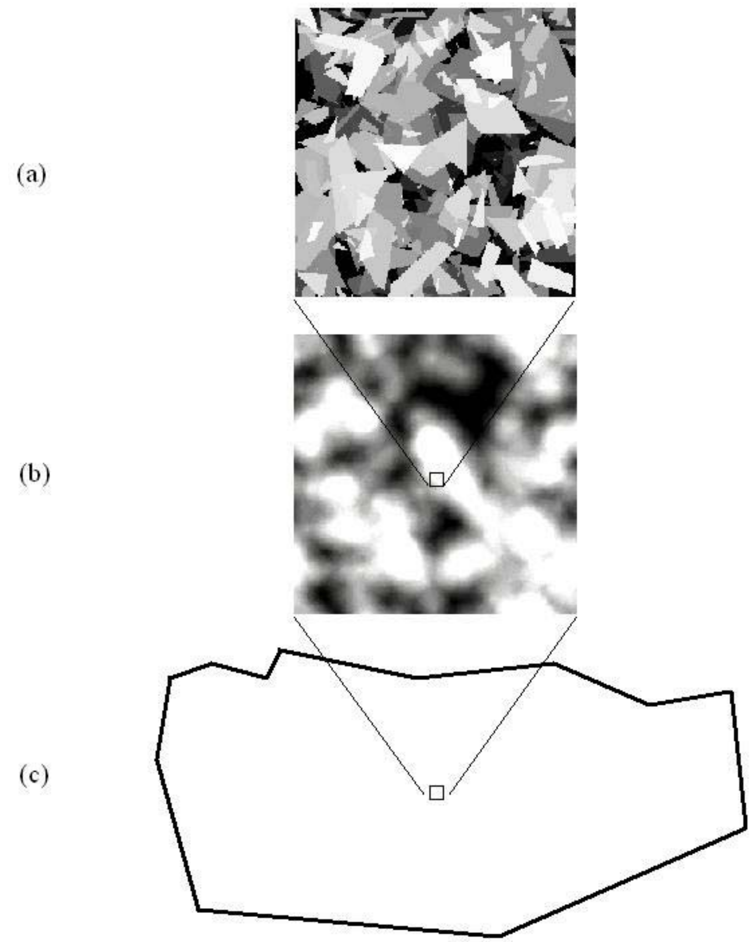
Continuum mechanics is the foundation of fluid dynamics and solid mechanics

Conventional assumption: smooth fields

Many materials/media have fractal geometries

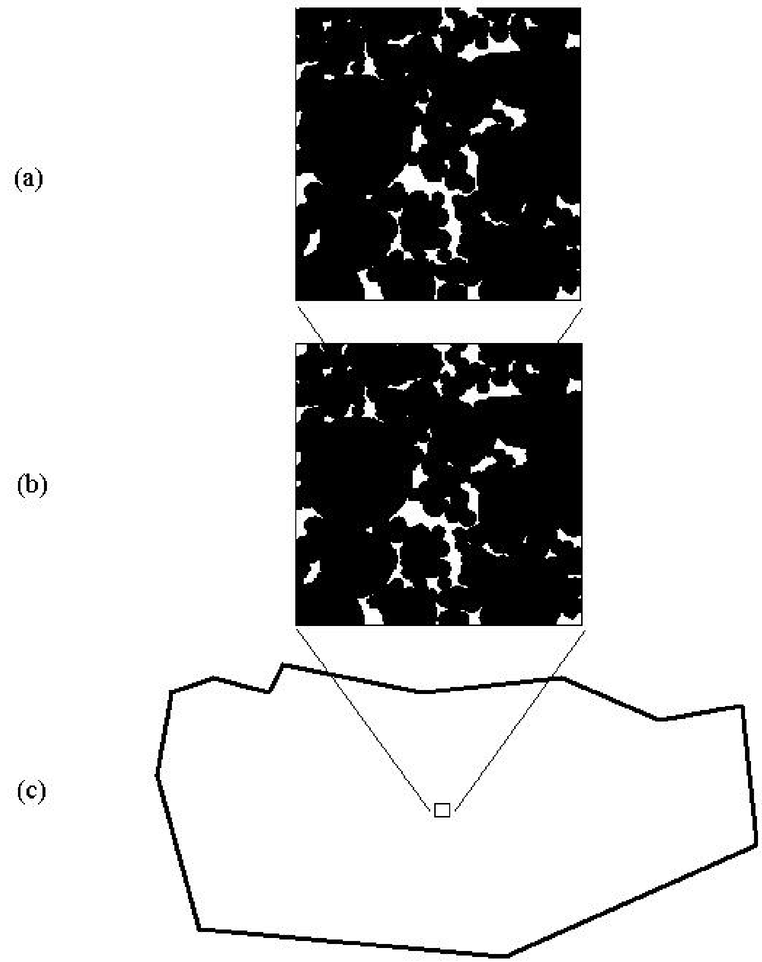
Need to extend continuum mechanics and thermodynamics to fractal (random) media

Want to construct models of anomalous response, e.g. viscoelasticity with fractional derivatives



Conventional continuum mechanics hinges on a **separation of scales**

$$d \ll L \ll L_{macro}$$



in fractal media:
no **separation of scales**
$d \ll L \ll L_{macro}$

thermomechanics = continuum mechanics + thermodynamics

Note: there are three basic thermomechanics formulations:

rational thermomechanics: stress, heat flux, internal energy, entropy are fundamental fields

extended rational thermomechanics: all field equations are hyperbolic

thermomechanics with internal variables: material response is described by two functionals: free energy ψ and dissipation function ϕ

2 Formulation and previous results

consider the medium $\mathcal{B} = \{B(\omega); \omega \in \Omega\}$ to have a random fractal geometry

mass m in each $B(\omega)$ [V. Tarasov, *Phys. Lett. A*, *Ann. Phys.* 2005]

$$m(R) = kR^D \quad D < 3 \tag{2.2}$$

$R =$ box size, $D =$ fractal dimension, and $k =$ proportionality constant

Focusing on porous media, (2.2) becomes

$$m_D(R) = m_0 \left(\frac{R}{R_p}\right)^D \tag{2.3}$$

$R_p =$ pore radius, $m_0 =$ mass at $R_p = R$

$D =$ fractal dimension of mass in a domain W

∂W has dimension d

... in general, d equals neither 2 nor $D - 1$.

$\Longrightarrow$ the conventional equation giving mass in a 3-D region W of volume V

$$m(W) = \int_W \rho(\mathbf{r})\, d^3\mathbf{r} \tag{2.4}$$

has to be generalized to

$$m_{3d}(W) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W \rho(\mathbf{r}) \left|\mathbf{r} - \mathbf{r}_0\right|^{D-3} d^3\mathbf{r} \tag{2.5}$$

if fractal medium is spatially homogeneous, $\rho(\mathbf{r}) = \rho_0 = const$,

$\Longrightarrow$ is replaced by a fractional integral

$$m_{3d}(W) = \rho_0 \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W |\mathbf{R}|^{D-3} d^3\mathbf{r}, \qquad \mathbf{R} = \mathbf{r} - \mathbf{r}_0 \qquad (2.7)$$

$\Longrightarrow$ interpretation of fractal medium as a continuum

use 'dimensional regularization' [J.C. Collins, *Renormalization*, 1984]

... need a reformulation of the Green-Gauss Theorem

$$\int_{\partial W} f v_k n_k dA_d = \int_W c_3^{-1} (D, R)\, div\, (c_2 (d, R)\, f\mathbf{v})\, dV_D \tag{2.8}$$

$f =$ arbitrary function, $\mathbf{v} =$ particle velocity, and

$$dA_d = c_2 (d, R)\, dA_2 \quad dV_D = c_3 (D, R)\, dV_3 \tag{2.9}$$

On account of (2.9), the LHS in (2.8) is a fractional integral, equal to a conventional integral $\int_{\partial W} c_2 (d, R)\, f\mathbf{v} dA_2$.

Similarly, RHS in (2.8) is a fractional integral, equal to a conventional integral $\int_W div\, (c_2 (d, R)\, A\mathbf{v})\, dV_3$.

$\Longrightarrow$ derivation of fractional-type balance equations of fractal media:

fractional equation of continuity:

$$\left(\frac{d}{dt}\right)_D \rho = -\rho \nabla_k^D v_k \tag{2.10}$$

fractional equation of balance of momentum density:

$$\rho \left(\frac{d}{dt}\right)_D v_k = \rho f_k + \nabla_l^D \sigma_{kl} \tag{2.11}$$

fractional equation of balance of energy density:

$$\rho \left(\frac{d}{dt}\right)_D u = c\,(D, d, R)\,\sigma_{kl} v_{k,l} - \nabla_k^D q_k \tag{2.12}$$

σ_{kl} = Cauchy stress (symmetric),

... and the following operators (or, generalized derivatives) are used

$$\nabla_k^D f = c_3(D,R)\frac{\partial}{\partial x_k}\left[c_2(d,R)\, f\right] \equiv c_3(D,R)\,\nabla_k\left[c_2(d,R)\, f\right]$$
$$\left(\frac{d}{dt}\right)_D f = \frac{\partial f}{\partial t} + c(D,d,R)\, v_k \frac{\partial f}{\partial x_k} \tag{2.13}$$

where

$$c(D,d,R) = |\mathbf{R}|^{d+1-D}\,\frac{2^{D-d-1}\Gamma(D/2)}{\Gamma(3/2)\,\Gamma(d/2)}$$
$$c_2(d,R) = |\mathbf{R}|^{D-3}\,\frac{2^{2-d}}{\Gamma(d/2)}$$
$$c_3(D,R) = |\mathbf{R}|^{2-d}\,\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \tag{2.14}$$
$$c(D,d,R) = c_3^{-1}(D,R)\, c_2(d,R)$$

Note: in a non-fractal medium ($D = 3$, $d = 2$) $c\left(D, d, R\right) = 1$, we recover conventional forms of local relations of continuum mechanics:

equation of continuity:

$$\frac{d}{dt}\rho = -\rho\nabla_k v_k \qquad (2.10')$$

equation of balance of momentum density:

$$\rho\frac{d}{dt}v_k = \rho f_k + \nabla_l \sigma_{kl} \qquad (2.11')$$

equation of balance of energy density:

$$\rho\frac{d}{dt}u = \sigma_{kl}v_{k,l} - \nabla_k q_k \qquad (2.12')$$

3 2nd law of thermodynamics in fractal media

begin with global form of 2nd law in V_D with Euclidean boundary ∂W

$$\dot{S} = \dot{S}^{(r)} + \dot{S}^{(i)} \quad \text{with } \dot{S}^{(r)} = \frac{\dot{Q}}{T}, \quad \dot{S}^{(i)} \geq 0 \tag{3.1}$$

$$\text{total entropy production rate} \equiv \dot{S} \geq \dot{S}^{(r)} \equiv \text{reversible entropy production rate} \tag{3.2}$$

$\Longrightarrow$

$$\frac{d}{dt}\int_W \rho s \; dV_D = \dot{S} \geq \dot{S}^{(r)} = -\int_{\partial W} \frac{q_k n_k}{T} dA_d \tag{3.3}$$

or

$$\int_W \rho \left(\frac{d}{dt}\right)_D s \, dV_D \geq - \int_W c_3^{-1} div \left(c_2 \frac{q_k}{T}\right) dV_D \qquad (3.4)$$

$\Longrightarrow$ local form of 2nd law

$$\rho \left(\frac{d}{dt}\right)_D s \geq -c_3^{-1} div \left(c_2 \frac{q_k}{T}\right) \qquad (3.5)$$

or, more explicitly,

$$\rho \left(\frac{d}{dt}\right)_D s \geq -c_3^{-1} \left[c_2 \frac{q_{k,k}}{T} - q_k \frac{c_{2,k} T - c_2 T_{,k}}{T^2}\right] \qquad (3.6)$$

Just like in thermomechanics of non-fractal bodies, introduce the rate of irreversible entropy production $\rho \, (d/dt)_D \, s^{(i)}$ which, in view of (3.6), gives

$$0 \leq \rho \left(\frac{d}{dt}\right)_D s^{(i)} = \rho \left(\frac{d}{dt}\right)_D s + c_3^{-1} c_2 \frac{q_{k,k}}{T} - c_3^{-1} q_k \frac{c_{2,k} T - c_2 T_{,k}}{T^2} \qquad (3.7)$$

Note: for a non-fractal body,

$$0 \leq \rho \frac{d}{dt} s^{(i)} = \rho \frac{d}{dt} s + \frac{q_{k,k}}{T} - q_k \frac{T_{,k}}{T^2} \tag{3.7'}$$

Recall $\psi = u - Ts$,

introduce a generalized deformation rate $\left[\left(\frac{d}{dt}\right)_D u_{(i}\right]_{,j)}$

use quasi-conservative stresses and entropy density

$$\sigma_{ij}^{(q)} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}} \qquad \beta_{ij}^{(q)} = \rho \frac{\partial \psi}{\partial \alpha_{ij}} \qquad s = -\frac{\partial \psi}{\partial T} \tag{3.12}$$

s.t.

$$\sigma_{ij} = \sigma_{ij}^{(q)} + \sigma_{ij}^{(d)}, \qquad \beta_{ij}^{(q)} = -\beta_{ij}^{(d)} \tag{3.13}$$

$\Longrightarrow$ generalization of the Clausius-Duhem inequality to fractal media*

$$0 \leq T\rho \left(\frac{d}{dt}\right)_D s^{(i)} = \sigma_{ij}^{(d)} \left[\left(\frac{d}{dt}\right)_D u_{(i}\right]_{,j)} + \beta_{ij}^{(d)} \left(\frac{d}{dt}\right)_D \alpha_{ij} - c\left(D, d, R\right) \frac{T_{,k} q_k}{T} \tag{3.18}$$

Note: if D is integer,

$$0 \leq T\rho \dot{s}^{(i)} = \sigma_{ij}^{(d)} d_{ij} + \beta_{ij}^{(d)} \dot{\alpha}_{ij} - \frac{T_{,k} q_k}{T} \tag{3.19}$$

*[M. Ostoja-Starzewski, *Zeit. Angew. Math. Phys.* 2007]

4 Thermodynamic Orthogonality

Observe: velocity $\mathbf{v} = \left\{ d_{ij},\ \dot{\alpha}_{ij},\ q_k \right\}$ is generalized to

$$\mathbf{v} = \left\{ \left[\left(\frac{d}{dt} \right)_D u_{(i} \right],_{j)},\ \left(\frac{d}{dt} \right)_D \alpha_{ij},\ q_k \right\}, \tag{4.1}$$

Introduce a dissipation functional in $\mathbf{V}$-space

$$\phi(\mathbf{v}) = T\rho \left(\frac{d}{dt} \right)_D s^{(i)} \geq 0. \tag{4.2}$$

and a dissipative force

$$\mathbf{A}^{(d)} = \left[\sigma_{ij}^{(d)},\ \beta_{ij}^{(d)},\ -c\,(D, d, R)\,\frac{T,_k}{T} \right]. \tag{4.3}$$

Postulate the Principle of Maximal Rate of Entropy Production in the $\mathbf{v}$-space: If $\mathbf{A}^{(d)}$ is prescribed, then the actual velocity vector $\mathbf{v}$ maximizes the disspation rate $L^{(d)} = \mathbf{A}^{(d)} \cdot \mathbf{v}$, subject to

$$\phi(\mathbf{v}) = \mathbf{A}^{(d)} \cdot \mathbf{v} \geq 0. \tag{4.4}$$

Equivalently, an extremum problem

$$\frac{\partial \phi}{\partial v_k} \left\{ A^{(d)} v_j - \mu \left[\boldsymbol{\Phi}(\mathbf{v}) - A_j^{(d)} d_j \right] \right\} = 0 \tag{4.5}$$

with Lagrangian multiplier μ, yields

$$A_i^{(d)} = \lambda \frac{\partial \boldsymbol{\Phi}}{\partial v_k} \qquad \lambda = \frac{\mu}{\mu + 1} \tag{4.6}$$

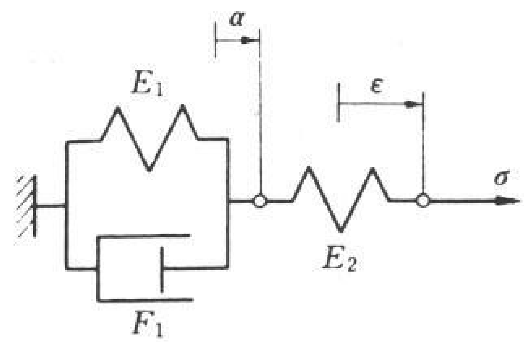
$\Longrightarrow$ constitutive laws of fractal media wtih two fundamental cases:

complex process:

$$\lambda = \phi \times \left[\frac{\partial \phi}{\partial \left[(d/dt)_D u_{(i} \right]_{,j)}} \left[\left(\frac{d}{dt} \right)_D u_{(i} \right]_{,j)} + \frac{\partial \phi}{\partial (d/dt)_D \alpha_{ij}} \left(\frac{d}{dt} \right)_D \alpha_{ij} + \frac{\partial \phi}{q_i} q_i \right]^{-1} \tag{4.7}$$

compound process:

$$
\begin{aligned}
\lambda^{(1)} &= \phi^{(1)} \left\{ \frac{\partial \phi^{(1)}}{\partial \left[(d/dt)_D \, u_{(i} \right] ,_{j)}} \left[(d/dt)_D \, u_{(i} \right] ,_{j)} \right\}^{-1} \\
\lambda^{(2)} &= \phi^{(2)} \left[\frac{\partial \phi^{(2)}}{\partial \, (d/dt)_D \, \alpha_{ij}} \, (d/dt)_D \, \alpha_{ij} \right]^{-1} \\
\lambda^{(1)} &= \phi^{(3)} \left[\frac{\partial \phi^{(3)}}{q_i} q_i \right]^{-1} \\
\phi \left\{ \left[\left(\frac{d}{dt} \right)_D u_{(i} \right] ,_{j)} , \right. & \left. \left(\frac{d}{dt} \right)_D \alpha_{ij}, \; q_k \right\} = \\
\phi^{(1)} \left\{ \left[\left(\frac{d}{dt} \right)_D u_{(i} \right] ,_{j)} \right\} &+ \phi^{(2)} \left[\left(\frac{d}{dt} \right)_D \alpha_{ij} \right] + \phi^{(3)} \left[q_k \right]
\end{aligned}
\tag{4.8}
$$



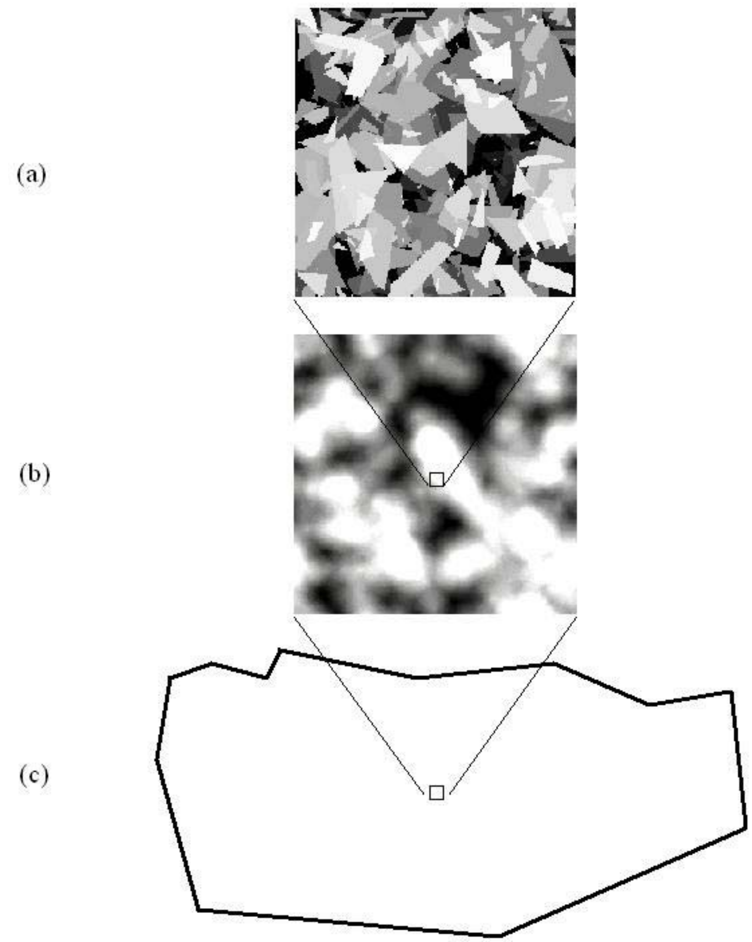
Kelvin element and spring in series

Example:
viscoelastic body of Zener type

$$\psi = \tfrac{1}{2}E_1\alpha^2 + \tfrac{1}{2}E_2\,(\varepsilon - \alpha)^2\,, \quad \phi = F_1\,[(d/dt)_D\,\alpha]^2 \tag{4.9}$$

$\Longrightarrow$

$$\sigma + \frac{F_1}{E_1 + E_2}\,(d/dt)_D\,\sigma = \frac{E_1E_2}{E_1 + E_2}\varepsilon + \frac{F_1E_2}{E_1 + E_2}\,(d/dt)_D\,\varepsilon \tag{4.10}$$



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5 Hill condition

stress and deformation rate fields as superpositions of means + fluctuations

$$\boldsymbol{\sigma}(\omega, \mathbf{x}) = \overline{\boldsymbol{\sigma}} + \boldsymbol{\sigma}'(\omega, \mathbf{x}) \qquad \mathbf{d}(\omega, \mathbf{x}) = \overline{\mathbf{d}} + \mathbf{d}'(\omega, \mathbf{x}) \tag{5.1}$$

$\Longrightarrow$ volume average of power density p in $B(\omega)$

$$\overline{p} \equiv \frac{1}{2V} \int_W \boldsymbol{\sigma}(\omega, \mathbf{x}) : \mathbf{d}(\omega, \mathbf{x}) dV_D = \frac{1}{2}\overline{\boldsymbol{\sigma} : \mathbf{d}} = \frac{1}{2}\overline{\boldsymbol{\sigma}} : \overline{\mathbf{d}} + \frac{1}{2}\overline{\boldsymbol{\sigma}' : \mathbf{d}'} \tag{5.2}$$

$\Longrightarrow$ Hill condition

$$\overline{\boldsymbol{\sigma} : \mathbf{d}} = \overline{\boldsymbol{\sigma}} : \overline{\mathbf{d}} \tag{5.3}$$

iff

$$\overline{\boldsymbol{\sigma}' : \mathbf{d}'} = 0 \tag{5.4}$$

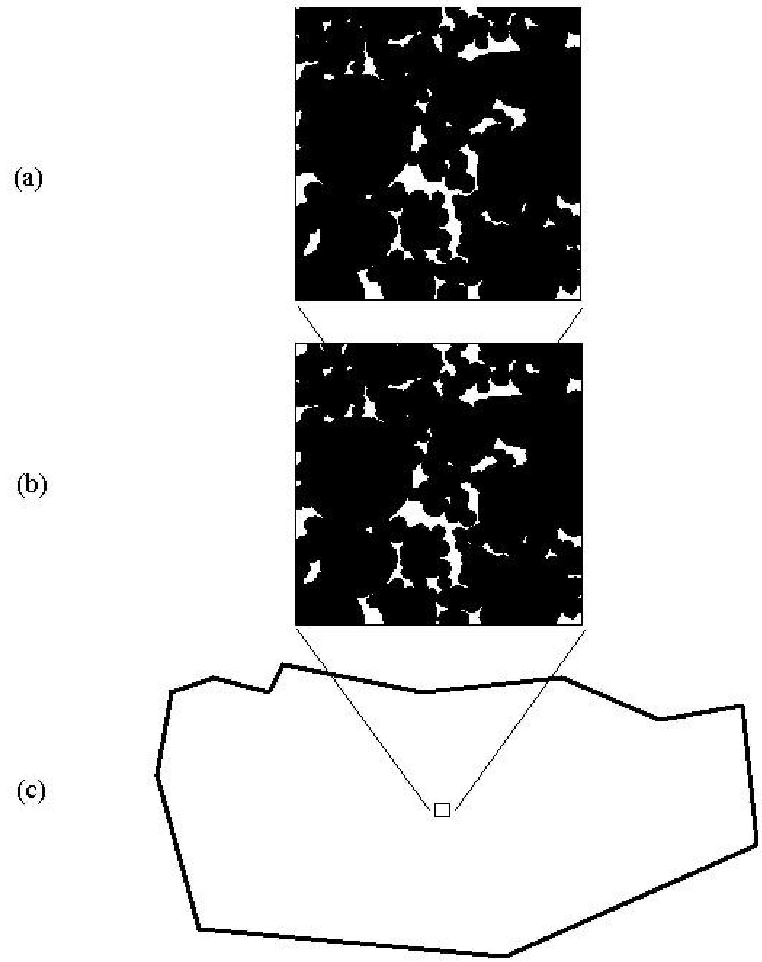
But, in a fractal medium:

$$\overline{c\,\boldsymbol{\sigma} : \mathbf{d}} = \overline{\boldsymbol{\sigma}} : \overline{\mathbf{d}} \tag{5.5}$$

$\Longrightarrow$

$$\overline{c\,\boldsymbol{\sigma}' : \mathbf{d}'} = 0 \quad \Leftrightarrow \quad \int_{\partial W} (\mathbf{t} - \overline{\boldsymbol{\sigma}} \cdot \mathbf{n})(\mathbf{v} - \overline{\mathbf{d}} \cdot \mathbf{x}) c_2 dA_2 = 0 \tag{5.8}$$

This dictates possible loadings on the boundary of a fractal body: uniform displacement, uniform traction and uniform displacement-traction (also called orthogonal-mixed) boundary conditions.



in fractal media:
no **separation of scales**
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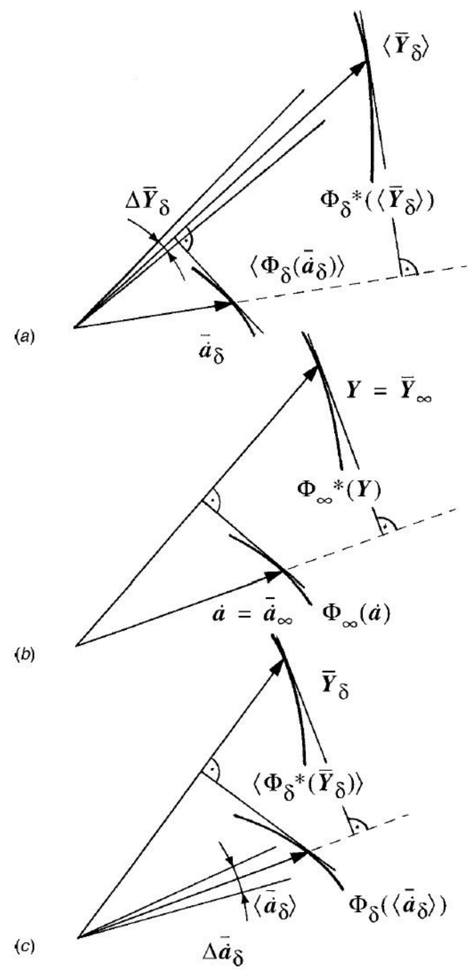
Legendre transforms



$\Longrightarrow$ Legendre transform pairs [9,10]:

$$\phi^* \left(\left\langle \overline{\mathbf{A}^{(d)}} \right\rangle \right) + \langle \phi \left(\overline{\mathbf{v}} \right) \rangle = \left\langle \overline{\mathbf{A}^{(d)}} \right\rangle \cdot \overline{\mathbf{v}} \tag{5.9}$$

and

$$\left\langle \phi^* \left(\overline{\mathbf{A}^{(d)}} \right) \right\rangle + \phi \left(\langle \overline{\mathbf{v}} \rangle \right) = \overline{\mathbf{A}^{(d)}} \cdot \langle \overline{\mathbf{v}} \rangle \tag{5.10}$$

The classical infinite size limit of mechanics of random media is unattainable unless one considers an upper cut-off to fractality, i.e. to the equation (2.3); then

$$\boldsymbol{\Phi}_{\delta}^{*eff} \left(\overline{\mathbf{Y}} \right) + \boldsymbol{\Phi}_{\delta}^{eff} \left(\bar{\dot{\mathbf{a}}} \right) = \overline{\mathbf{Y}} \cdot \bar{\dot{\mathbf{a}}} \tag{5.11}$$

which is a Legendre transform for a so-called representative volume element (RVE) of the continuum.

6 Thermoelasticity of fractal media*

Duhamel's equation of heat conduction

$$\rho c \frac{T}{T_0} \dot{T} = -\left(3\lambda + 2\mu\right) \kappa T \dot{\varepsilon}_{(1)} + \nabla_k \left(K \frac{\partial T}{\partial x_k} \right) \qquad (6.1)$$

is generalized to

$$\rho c \frac{T}{T_0} \dot{T} = -\left(3\lambda + 2\mu\right) \kappa T \dot{\varepsilon}_{(1)} + \nabla_k^D \left(K \frac{\partial T}{\partial x_k} \right) \qquad (6.1')$$

*[M. Ostoja-Starzewski, *J. Thermal Stresses*, 2007]

Heat conduction in a fractal rigid conductor

$$\rho c \frac{T}{T_0} \dot{T} = \nabla_k \left(K \frac{\partial T}{\partial x_k} \right) \tag{6.2}$$

is generalized to

$$\rho c \frac{T}{T_0} \dot{T} = \nabla_k^D \left(K \frac{\partial T}{\partial x_k} \right) \tag{6.2'}$$

7 Closure

- continuum mechanics of fractal media extended to dissipative field phenomena

- use thermomechanics with internal variables

- generalization of the Clausius-Duhem inequality

- admissible boundary loadings ensuring homogenization are found

- thermoelasticity and heat conduction extended to fractal media