

Robust and Efficient Interaction with Complex Systems*

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Abstract - *Low-dimensional chaotic map dynamics has been successfully used to predict the dynamics of high-dimensional systems far from equilibrium. For control, low-dimensional models can be used only if the control force is very small, otherwise hidden degrees of freedom may become excited. We study the control of chaotic map dynamics with extremely small forcing functions. We find that the smallest forcing function, which is called a resonant forcing function, echoes the natural dynamics of the system. This means, when the natural dynamics is irregular, the optimal forcing function is irregular too. If the natural dynamics contains a certain periodicity, the optimal forcing function contains that periodicity too. We show that such controls are effective even if the system has hidden degrees of freedom and if the probes of the control system have a low resolution. Further we show that resonant forcing functions of chaotic systems decrease exponentially, where the rate equals the Liapunov exponent of the unperturbed system. We apply resonant forcing functions for efficient control of chaotic systems and for system identification.*

Keywords: Chaos, resonance, optimal control, system identification.

1 Introduction

Galileo Galilei introduced the concept of resonance in “Discorsi a Dimostrazioni Matematiche” (1638) [22]. He uses the term resonance (“risonanza”) to describe the response of musical instruments to sound waves (lat. “resonare” = to echo). The response is largest if the frequency of the sound wave matches the natural frequency of the musical instrument. In the 18th century Huygens [1] discovered that two weakly coupled nonlinear oscillators, such as two mechanical clocks mounted on the same wall, show a strong interaction and tend to synchronize if their speed differs only slightly. In the synchronized state the energy exchange of coupled oscillators is comparatively large [10, 11] even if the coupling is weak [13, 16, 18, 20]. Haken showed that the dynamics of high-dimensional systems, including LASERS can often be predicted by low-dimensional

models, and that the resonances of the low dimensional model match the resonances of the high-dimensional system [14].

Resonance is a fundamental paradigm in physical sciences and engineering. In quantum physics, Bohr [3] introduced the term resonance frequency for the radiation frequency of a decaying quantum system. Breit and Wigner [25] used the concept of resonance to describe a sharp peak in the cross section for certain scattering events of particles in nuclear physics. In engineering and applied physics, most textbooks and review papers [5, 6, 9, 12, 23] define resonance by the largest response of a sinusoidal driven damped linear oscillator. At resonance the driving frequency of an external, sinusoidal force matches the natural frequency of the oscillator. A resonant forcing causes a large energy transfer to the oscillator. Technical applications of this concept are mostly in mechanical engineering and electrical engineering, e.g. RLC circuits in which the power consumption [5] of the circuit reaches its maximum at resonance. All spectroscopic instruments are based on resonant driving forces since this yields a large signal to noise ratio for harmonic and weakly nonlinear oscillators.

For highly nonlinear oscillators, the response to resonant sinusoidal driving forces is largest compared to other sinusoidal forcing functions. However it is rather small and complicated compared to the response of linear oscillators [2, 10,11]. It can be determined with the Lindstedt-Poincare method [20], Guckenheimer and Holmes’s averaging methodologies [13], Lichtenberg and Lieberman’s secular perturbation approach [18], or Nayfeh and Mook’s multiple scales methodologies [20]. Since the response of nonlinear oscillators to sinusoidal resonant forcing functions is rather small, the resulting signal-to-noise ratio for spectroscopic instruments is also small. Wargitsch and Hubler [15, 24] and other [17, 21] have shown, that a special class of aperiodic driving forces can achieve a large energy transfer to a nonlinear oscillator. Such non-sinusoidal resonant forcing functions yield a high signal-to-noise ratio, which can be used for system identification with general resonance spectroscopy

[8]. The non-sinusoidal resonant forcing functions have the same dynamics as the time reflected transient dynamics of the unperturbed system [24].

In this paper, we present a methodology to determine resonant forcing functions for chaotic systems. In section 2 we show analytically that optimal driving forces for chaotic map dynamics are closely related to the unperturbed dynamics of the system. Further, we show that resonant forcing functions decrease exponentially, where the rate is equal to the Liapunov exponent of the unperturbed system (Section 2.2). In Section 2.3 we show that resonant forcing functions yield a large response, even if the initial condition of the system is not exactly known. In Section 2.4 we illustrate with a specific example that hidden variables are not excited if the magnitude of the forcing function is small. In Section 3 we apply resonant forcing functions for resonance spectroscopy. We compare spectroscopic methods based on sinusoidal forcing functions with spectroscopic methods based on aperiodic, resonant forcing functions..

2 Resonant stimulation of chaotic map dynamics

We consider systems where x_n is the state at time step $n=0, 1, 2, \dots, N-1$ and the dynamics is given by the mapping function f :

$$x_{n+1} = f(x_n) + F_n \quad (1)$$

where F_n is the forcing at time step n . The magnitude of the forcing function F is defined as

$$F^2 = \sum_n F_n^2. \quad (2)$$

The final response R to the forcing is defined as

$$R = (x_N - y_N)^2 \quad (3)$$

where $y_{n+1} = f(y_n)$ is the unperturbed dynamics and $y_0 = x_0$.

2.1 Optimal forcing functions

We use the calculus of variations [7, 19] with Lagrange function L to determine the forcing function with yields the largest final response R , where L is

$$L = (x_N - y_N)^2 +$$

$$+ \nu \sum_n F_n^2 + \sum_n \mu_n (x_{n+1} - f(x_n) - F_n) \quad (4)$$

and where ν , and μ_n , are Lagrange multipliers. If we eliminate the Langrange multipliers from the corresponding equations of motion [4] and assume that f is differentiable at all x_n , the dynamics of the optimal forcing is given by:

$$F_n = F_{n-1} / f'(x_n) \quad (5)$$

and $(x_N - y_N) + \nu F_{N-1} = 0$, where $F_0^2 = F^2 / (1 + \sum_m \prod_k f'(x_k)^2)$, $k=1, 2, \dots, m$, and $m=1, 2, \dots, N-1$. We call the optimal forcing function, given by Eq. 5, a resonant forcing.

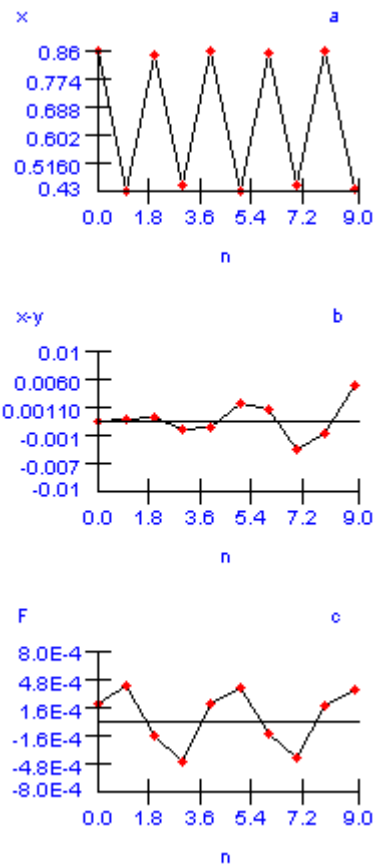


Figure 1. Resonant forcing of a period 4 logistic map dynamics. The forced dynamics x_n (a), the response $R_n = x_n - y_n$ (b), and the resonant forcing F_n (c) contain period 4 also. ($F=0.001$)

2.2 Approximation of the optimal forcing function

If the magnitude of the forcing F is small, we can approximate x_n in Eq. 5 by the unperturbed dynamics y_n and obtain:

$$F_{n+1} = F_0 / \prod_j f'(y_j) \quad (6)$$

where $j=1, 2, 3, \dots, n$, $F_0^2 = F^2 / (1 + \sum_m \prod_k f'(y_k)^{-2})$, $k=1, 2, \dots, m$, and $m=1, 2, \dots, N-1$.

where $\lambda = \lim_{N \rightarrow \infty} (1/N) \sum_{m=1}^N \ln(f'(y_m))$, $m=1, 2, 3, \dots, N$ and $F_0^2 = F^2 (1 - e^{-2\lambda}) / (1 - e^{-2\lambda N})$. If the real part of the Liapunov exponent is positive the unperturbed dynamics is chaotic, otherwise not.

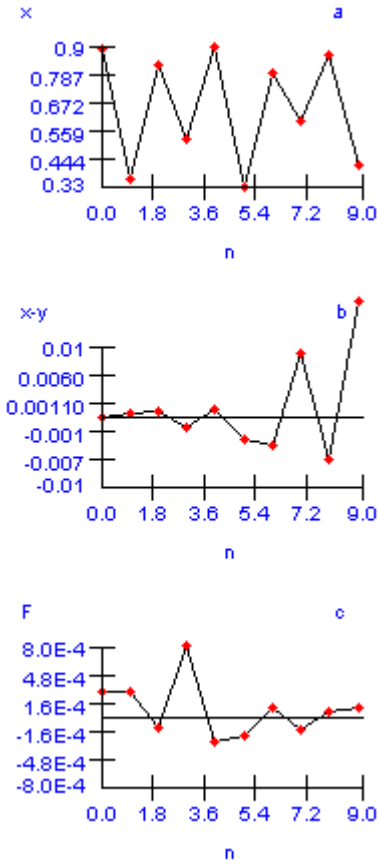


Figure 2. Resonant forcing of a chaotic logistic map dynamics. The forced dynamics x_n (a), the response $R_n = x_n - y_n$ (b), and the resonant forcing F_n (c) are irregular. ($F=0.001$)

Eq. 6 shows that the resonant forcing can be computed from the unperturbed dynamics with a simple algebraic transformation. Hence when the unperturbed dynamics is irregular, the optimal forcing is irregular too. If the unperturbed dynamics is periodic, the resonant forcing functions contain the same periodicity. Fig. 1 shows a resonant forcing of a logistic map dynamics $f(x_n) = a x_n (1 - x_n)$, with a period 4 dynamics ($a=3.46$). The resonant forcing function is computed with Eq. 5, however forcing functions computed with Eq. 6 agree within a few percent. Fig. 2 shows a resonant forcing of a chaotic logistic map dynamics $x_{n+1} = a x_n (1 - x_n)$, where $a=3.6$.

For large n the complex Liapunov exponent can approximate the product in Eq. 6:

$$F_n = F_0 e^{-n\lambda} \quad (7)$$

Hence we conclude from Eq. 7 that the magnitude of the resonant forcing function for chaotic map dynamics decreases exponentially. The rate of the exponential decrease is equal to the Liapunov exponent of the unperturbed dynamics (see Fig. 2). In contrast, resonant forcing functions increase exponentially if the unperturbed system is near a stable periodic attractor (see Fig. 1).

2.3 Sensitivity to the initial conditions

Resonant forcing functions in Eq. 6 depend on the initial state of the system y_0 . In this section we study the size of the response as function of the uncertainty in the initial condition $d = x_0 - y_0$. Numerical simulations suggest that a small uncertainty has a small effect on the magnitude of the response.

Fig. 3 shows the final response R as a function of the uncertainty in the initial condition for a logistic map dynamics ($a=3.6$). In this case uncertainties less than 1% of the system size have a small effect on the response.

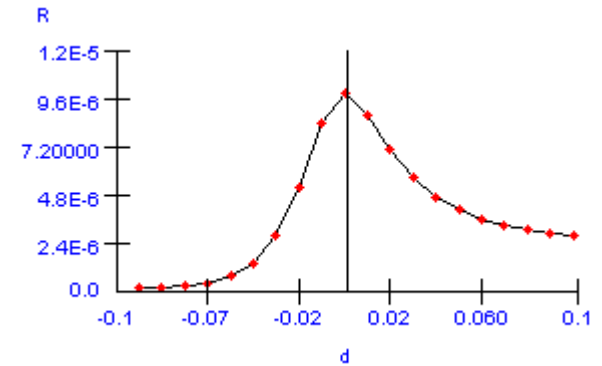


Figure 3. The final response R as a function of the uncertainty in the initial condition d for a chaotic logistic map dynamics. ($N=4$, $x_0=0.37$, $a=3.6$, $F=0.001$)

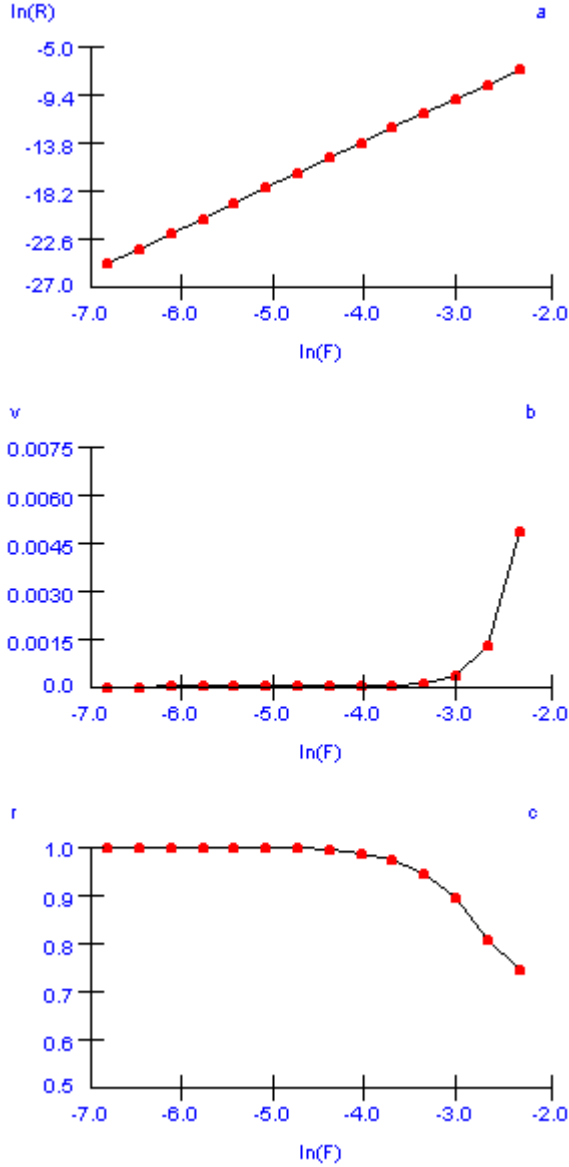


Figure 4. The natural logarithm of the final response $\ln(R)$ versus the natural logarithm of the magnitude of the forcing function $\ln(F)$ (a); the magnitude of the hidden variable $v=|v_N|$ versus the natural logarithm of the magnitude of the forcing function $\ln(F)$ (b); the ratio r between the final response R of the two dimensional chaotic system and the one-dimensional chaotic system versus the natural logarithm of the magnitude of the forcing function. The resonant forcing function is computed with the one-dimensional model. ($x_0=0.37$, $a=3.6$, $F=0.001$)

2.4 Excitation of hidden variables

One of the main reasons to use resonant forcing functions for control is to avoid interference from hidden variables. In this section we replace the system given by Eq. 1 with a system with a state x'_n and a hidden variable v_n :

$$x'_{n+1} = f(x'_n) + v_n + F_n \quad (8)$$

$$v_{n+1} = 0.5 v_n + (10F_n)^2$$

This model assumes that the hidden variable is in lowest order only driven by the forcing function, whereas the forcing function and the hidden variable drives the main variable.

If F_n is small, then v_n is small. In this case Eq. 1 and Eq. 8 are effectively equivalent. In the following we use the one-dimensional model (Eq. 1) to compute the optimal forcing function F_n (Eq. 5) and determine the final response $R' = (x'_N - y_N)^2$ of the two-dimensional model (Eq. 8). To compare the final response of the one-dimensional system we compute the ratio r :

$$r = R'/R \quad (9)$$

Fig. 4 a shows that the response R is proportional to F^2 , $R \sim F^2$. For large forcing functions the hidden variable becomes excited (Fig. 4 b) and the response of the two dimensional systems is less than the response we expect from a one-dimensional system. Hence hidden variables have an impact on the response if the forcing function is large.

3 Resonance Spectroscopy

3.1 Spectroscopy

Here we consider systems where the mapping function f contains an unknown parameter a :

$$x_{n+1}(a, b) = f(x_n, a) + F_n(b) + \rho_n \quad (10)$$

where F_n is the forcing at time step n which may depend on a model parameter b . ρ_n is additive random noise at time step n . The response is predicted with a model:

$$X_{n+1} = f(X_n, b) + F_n(b) \quad (11)$$

where X_n is the state of the model system at time n . If the model is exact ($a=b$, $x_0 = X_0$ and $\rho_n=0$), we expect that the deviation D between the observed response and the predicted response:

$$D(b) = (x_N(a, b) - X_N(b))^2 \quad (12)$$

is zero. This can be used for system identification:

$$a \in \{b_i | D(b_i) = 0\} \quad (13)$$

where b_i are the roots of D . System identification is unique if $D(b)$ has only one root. The number of roots depends on the forcing function. Fig. 5 shows the deviation $D(b)$ for a chaotic logistic map dynamics $f(x_n, a) = a x_n (1 - x_n)$, where $a=3.6$ and a periodic forcing function $F_n = 0.0005 * (-1)^n$.

3.2 Resonance spectroscopy

In the section we consider a set of forcing functions, which contains the resonant forcing function. All forcing functions in the set have the same magnitude F , and each forcing function is the resonant forcing function for a particular model with model parameter b . Hence, $F_n(b)$ is computed with Eq. 5:

$$F_n(b) = F_{n-1}(b)/f'(X_n, b) \quad (14)$$

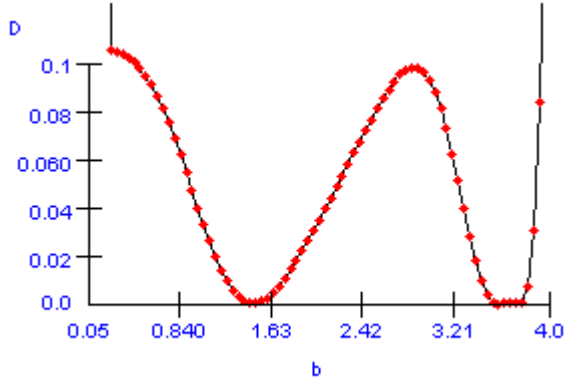


Figure 5. The deviation D between model and system as a function of the model parameter b for a chaotic logistic map dynamics. ($N=4$, $x_0=0.37$, $a=3.6$, $F=0.001$). $D(b)$ has a unique root at $b=a$.

or one of the approximations discussed in Sec. 2.2. The final response versus the model parameter b is called the resonance curve $R(b)$:

$$R(b) = (x_N(a, b) - y_N)^2 \quad (14)$$

where $x_N(a, b)$ is given by Eq. 10 and $y_{n+1} = f(y_n, a)$ and $y_0 = x_0$ is the unperturbed dynamics. We expect that the response is largest for the resonant forcing, since resonant forcing functions are optimized with the calculus of variations (Eq. 4). Hence the resonance curve has an absolute maximum at $a=b_{max}$:

$$R(b_{max}) \geq R(b) \text{ for all } b. \quad (15)$$

There are possibly several b values where $R(b) = R(b_{max})$, but we didn't observe this in our simulations. Fig. 6 shows the resonance curve for a chaotic logistic map dynamics. The resonance curve has its absolute maximum at $a=b$.

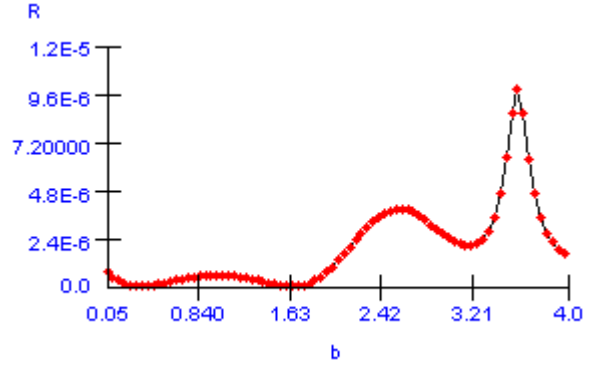


Figure 6. The response R versus the model parameter b for a chaotic logistic map dynamics. The resonance curve $R(b)$ has its absolute maximum at $b=a$. ($N=4$, $x_0=0.37$, $a=3.6$, $F=0.001$).

4 Conclusions

We show that resonant forcing functions of chaotic systems are closely related to their natural dynamics (Eq. 6). This means, when the natural dynamics is irregular, the optimal forcing function is irregular too, and if the natural dynamics contains a certain periodicity, resonant forcing functions contain the same periodicity (Fig. 1 and 2). We find that resonant forcing functions of chaotic systems decrease exponentially, where the rate equals the Liapunov exponent of the unperturbed system. Resonant forcing functions depend on the initial state of the systems, however for the logistic map, for example, the dependence on the initial condition is smooth (Fig. 3). Further we have shown that low-dimensional models can be used for controlling multi-dimensional systems if the control force is very small (Fig. 4).

We found that the resonance curve $R(b)$ for chaotic map dynamics has an absolute maximum when the model parameter b matches the system parameter a , i.e. $a=b$ (Eq. 15). We show that resonant forcing functions of chaotic systems can be used for system identification.

5 References

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